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Slip flow of a viscous fluid past an inclined flat plate

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SUMMARY

Viscous flow of a slightly rarefied gas past a flat plate inclined to a uniform stream is studied analytically on the basis of the Oseen equation. A set of singular integral equations for the distribution of Oseenlets along the surface of the plate is derived from the slip boundary condition and solved by a method of matched asymptotic expansions. The drag and lift forces acting on the plate are calculated correctly to the order of the Knudsen number k. The results show that the lift coefficient increases owing to the slip by an amount of $O(k\,|\ln k\,|)$ while the drag decreases.

1. Introduction

Rarefaction of a gas has a large influence on flows at small Reynolds numbers, even though the flow velocity is much smaller than the speed of sound. As the Reynolds number decreases at a fixed Mach number, the nature of flow changes from continuum to free molecule. Such transition was investigated experimentally for the flows past a cynlinder and a flat plate [1]. Some attempts to illustrate the transition have been made on the theoretical side. When the molecular mean free path of the gas is small but not negligible compared with a characteristic dimension of the flow, the rarefaction effect reduces to a slip of macroscopic (continuum) flow on a solid surface. According to the kinetic theory, the magnitude of the slip velocity is expressed as the product of the mean free path and the gradient of tangential velocity [2]. Analyses of viscous flows taking the slip boundary condition into account were done for the cases of a circular or elliptic cylinder [3, 4, 5] and a flat plate [6].

Departure due to slip from the classical no-slip solution can be calculated by a simple perturbation method if the solid body is of smooth shape. On the other hand, such a method is not applicable to the case of a flat plate since the velocity gradient becomes infinite at its edges in the no-slip solution. Slip modifies completely the flow structure given by the no-slip solution near the edges of the plate. Flow around the leading edge of a semi-infinite flat plate admitting slip was examined by Laurmann [7] using the Oseen equation and by van de Vooren and Veldman [8] using the Navier-Stokes equation. Their results showed that the slip velocity at the edge takes a finite value proportional to the square root of the mean free path. Tamada and Miura [6] solved the Oseen equation subject to the slip condition for a flat plate of finite length placed at zero incidence. Slip of the flow was analyzed not only in the vicinities of the leading and trailing edges but also over the central part of the plate. It was found that the slip gives rise to reduction of the drag force acting on the plate by an amount of $O(k\,|\ln k\,|)$, where k is the Knudsen number, the ratio of the mean free path to the length of the plate.

The present paper is concerned with slip flow past a plate of finite length when it is inclined to a uniform stream. Two-dimensional motion of the gas at small Reynolds numbers is assumed to obey the Oseen linearized equation. Then, the problem is reduced to solving a set of singular integral equations for the distributions of dragging and lifting Oseenlets along the plate. An approximate method of solution similar to that developed in [6] is applied and the formulae for the drag and lift forces are obtained. The results indicate a transition from the no-slip continuum flow.

2. Formulation of the problem

Consider a flat plate placed at an angle of attack α in an otherwise uniform stream of a gas with velocity U. Let Uu and Uv be the velocity components of perturbation from the uniform flow in the directions of the free stream and normal to it, respectively. We take Cartesian coordinates (x, y) normalized by the length of the plate l with the x-axis taken along the plate (see Figure 1). The solution satisfying the boundary condition that the perturbation vanishes at infinity can be constructed in terms of a suitable distribution along the plate of the fundamental solution known as Oseenlet [9]:

$$u - iv = \frac{1}{2\pi}\int_0^1 \left[\frac{2}{R\tilde{r}} e^{-i\tilde{\theta}} (f_1(\xi) - if_2(\xi)) - \exp\left(\frac{R}{2}\tilde{X}\right)\left\{K_0\left(\frac{R}{2}\tilde{r}\right)(f_1(\xi) + if_2(\xi)) + K_1\left(\frac{R}{2}\tilde{r}\right) e^{-i\tilde{\theta}}(f_1(\xi) - if_2(\xi))\right\}\right] d\xi, \tag{1}$$

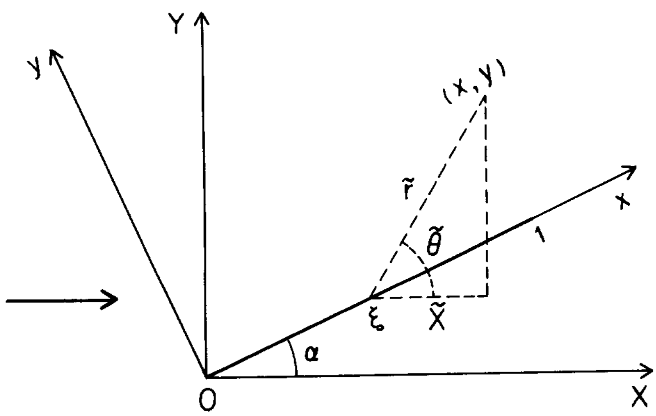

Figure 1. Configuration.

where $R = Ul/\nu$ is the Reynolds number, ν the kinematic viscosity,

$$\tilde{r} = \{(x-\xi)^2 + y^2\}^{1/2}, \quad \tilde{\theta} = \alpha + \tan^{-1}\{y/(x-\xi)\}, \quad \tilde{X} = (x-\xi)\cos\alpha - y\sin\alpha$$

and K_0 and K_1 are the modified Bessel functions. Since we are concerned with the case when the Reynolds number is considerably smaller than unity, this expression for the perturbation velocity can be approximated in the neighbourhood of the plate as follows:

$$u - iv = \frac{1}{2\pi}\int_0^1 \left[\left\{\frac{1}{2}\ln((x-\xi)^2 + y^2) - \ln\frac{4}{R} + \gamma\right\}\{f_1(\xi) + if_2(\xi)\}\right.$$
$$-\frac{(x-\xi)\cos\alpha - y\sin\alpha}{(x-\xi)^2 + y^2}\{((x-\xi)\cos\alpha - y\sin\alpha) - i((x-\xi)\sin\alpha + y\cos\alpha)\}$$
$$\left.\{f_1(\xi) - if_2(\xi)\}\right]d\xi, \tag{2}$$

where $\gamma = 0.5772\ldots$ is the Euler constant. It should be noted that this is a solution to Stokes approximation neglecting the convective effect.

The slip boundary condition on the surface of the plate is given by

$$(1+u)\cos\alpha + v\sin\alpha = k\left(\frac{\partial u}{\partial y}\cos\alpha + \frac{\partial v}{\partial y}\sin\alpha\right)\operatorname{sgn}(y),$$
$$v\cos\alpha - (1+u)\sin\alpha = 0 \quad \text{at} \quad y = \pm 0 \quad (0<x<1). \tag{3}$$

Substitution of (2) into (3) leads to a set of integral equations for the distribution functions $f_1(x)$ and $f_2(x)$:

$$1 + \frac{1}{2\pi}\int_0^1 \left\{\left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma - 1\right)f_1(\xi) - \left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma\right)f_2(\xi)\tan\alpha\right\}d\xi$$
$$= k\{f_1(x) - f_2(x)\tan\alpha\},$$
$$1 + \frac{1}{2\pi}\int_0^1 \left\{\left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma\right)f_1(\xi) + \left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma + 1\right)f_2(\xi)\cot\alpha\right\}d\xi$$
$$= 0 \quad (0<x<1). \tag{4}$$

The integral equations (4) are not amenable to a simple procedure and we shall seek a solution for small k in an approximate way. We consider separately narrow regions near the leading and trailing edges, where the slip produces a large effect, and, a central part of the plate, where deviation from the no-slip solution is small. Matching the solutions valid in these regions makes it possible to get an overall solution.

It is convenient to use the stretched variables given by

$$t = k^{-1}x, \qquad \tau = k^{-1}\xi \tag{5}$$

for discussing the behaviour of the solution in the vicinity of the leading edge. We differentiate (4) with respect to x and introduce the new variables. Replacing k^{-1} approximately by infinity yields the simplified forms of equations valid in the narrow region near the leading edge:

$$\begin{aligned} &⨍_0^\infty \frac{1}{t-\tau}\{g_1(\tau) - g_2(\tau)\tan\alpha\}d\tau = 2\pi\frac{d}{dt}\{g_1(t) - g_2(t)\tan\alpha\}, \\ &⨍_0^\infty \frac{1}{t-\tau}\{g_1(\tau) + g_2(\tau)\cot\alpha\}d\tau = 0 \qquad (t > 0), \end{aligned} \tag{6}$$

where Cauchy principal values are taken for the integrals and

$$g_j(t) = f_j(kt) \qquad (j = 1, 2). \tag{7}$$

It should be noted that direct application of the approximation made above to (4) without differentiation would bring about a meaningless result of divergence of the integrals since both $g_1(t)$ and $g_2(t)$ change like $t^{-1/2}$ for $t \gg 1$ as is seen later.

3. Approximate solution

The integral equations (6) can be solved by means of the Wiener-Hopf technique. The leading edge solution valid for $x = O(k)$ is (see Appendix A)

$$\begin{aligned} f_{1e}(x) &= \frac{A\cos^2\alpha}{\pi}\int_0^\infty \{p(p+\tfrac{1}{2})\}^{-1/2}\sin\left\{\frac{x}{k}p - \frac{1}{2\pi}P(p) + \frac{\pi}{4}\right\}dp \\ &\quad + B\sin^2\alpha\left(\frac{k}{\pi x}\right)^{1/2}, \\ f_{2e}(x) &= -\frac{A\sin\alpha\cos\alpha}{\pi}\int_0^\infty \{p(p+\tfrac{1}{2})\}^{-1/2}\sin\left\{\frac{x}{k}p - \frac{1}{2\pi}P(p) + \frac{\pi}{4}\right\}dp \\ &\quad + B\sin\alpha\cos\alpha\left(\frac{k}{\pi x}\right)^{1/2}, \end{aligned} \tag{8}$$

where A and B are certain constants and

$$P(p) = \int_0^p \frac{\ln(2s)}{s^2 - \frac{1}{4}} ds. \tag{9}$$

The factors A and B are determined by the principle that this solution should match the solution valid in the central region of the plate. The asymptotic form of (8) for $x \gg k$ is

$$\begin{aligned} f_{1e}(x) &\sim (2^{1/2} A \cos^2\alpha + B \sin^2\alpha)(k/\pi x)^{1/2}, \\ f_{2e}(x) &\sim (-2^{1/2} A + B) \sin\alpha \cos\alpha (k/\pi x)^{1/2}. \end{aligned} \tag{10}$$

Within the Stokes approximation, the flow velocity is symmetric with respect to the center of the plate even under the slip boundary condition while the perturbation pressure is anti-symmetric. Accordingly, the distribution functions f_1 and f_2, which represent the local force acting on the plate, are also symmetric and the trailing-edge solution is given by $f_{1e}(1-x)$ and $f_{2e}(1-x)$.

Slip of the flow is supposed to be small over most of the plate except for narrow edge-regions. Hence we may take the no-slip solution as the main solution to the first approximation valid in this wide domain. The no-slip solution of (4) with $k = 0$ has been obtained by Miyagi [10] in the form:

$$\begin{aligned} f_{1m}(x) &= 2(T - \cos^2\alpha)(T^2 - \cos^2\alpha)^{-1} \{x(1-x)\}^{-1/2}, \\ f_{2m}(x) &= 2 \sin\alpha \cos\alpha (T^2 - \cos^2\alpha)^{-1} \{x(1-x)\}^{-1/2}, \end{aligned} \tag{11}$$

where

$$T = \ln(16/R) - \gamma. \tag{12}$$

The leading-edge solution should agree with the main solution in an overlap domain of $1 \gg x \gg k$. From comparison of (10) with (11) for small x, we have

$$\begin{aligned} A &= (T-1)(T^2 - \cos^2\alpha)^{-1} (2\pi/k)^{1/2}, \\ B &= 2T(T^2 - \cos^2\alpha)^{-1} (\pi/k)^{1/2}. \end{aligned} \tag{13}$$

A solution which holds uniformly over the whole of the plate can be composed in terms of the leading edge, main and trailing edge solutions [11]. The overall solution to the first order is

$$\begin{aligned} f_{1u}(x) &= f_{1e}(x) + f_{1m}(x) + f_{1e}(1-x) - 2(T - \cos^2\alpha)(T^2 - \cos^2\alpha)^{-1} \\ &\quad \{x^{-1/2} + (1-x)^{-1/2}\}, \\ f_{2u}(x) &= f_{2e}(x) + f_{2m}(x) + f_{2e}(1-x) - 2 \sin\alpha \cos\alpha (T^2 - \cos^2\alpha)^{-1} \\ &\quad \{x^{-1/2} + (1-x)^{-1/2}\}. \end{aligned} \tag{14}$$

Our attention has been directed to the marked slip effect near the edges of the plate. The slip over the central part of the plate is small locally, but it may produce a significant correction to the force acting on the plate since the central region is much wider than the edge regions. We must examine small perturbation to the main solution (11). The singularities of (11) at the edges of the plate, however, make a standard perturbation analysis inapplicable. In order to avoid this difficulty, we put

$$f_j(x) = f_{ju}(x) + \hat{f}_j(x) \qquad (\hat{f}_j \ll f_{ju}, j = 1, 2). \tag{15}$$

Substituting (15) into (4), we get a set of integral equations for $\hat{f}_1(x)$ and $\hat{f}_2(x)$ to $O(k)$ (see Appendix B):

$$\frac{1}{2\pi}\int_0^1 \left\{\left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma - 1\right)\hat{f}_1(\xi) - \left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma\right)\hat{f}_2(\xi)\tan\alpha\right\} d\xi$$

$$= 2k\frac{T-1}{T^2 - \cos^2\alpha}\left[\left\{\frac{1}{[x(1-x)]^{1/2}} - \frac{1}{x^{1/2}} - \frac{1}{(1-x)^{1/2}}\right\} + \frac{1}{\pi^2}\left(\ln\frac{2}{k} + \gamma + 1\right)\right.$$

$$\left\{\ln(1-x) + \frac{1}{x^{1/2}}\ln\frac{1+x^{1/2}}{1-x^{1/2}} + \ln x + \frac{1}{(1-x)^{1/2}}\ln\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}}\right\} - \frac{4}{\pi^2}\int_0^1$$

$$\left.\left\{\frac{1}{1-x\eta^2} + \frac{1}{1-(1-x)\eta^2}\right\}\ln\eta\, d\eta - \frac{2}{\pi^2}\left(\ln\frac{4}{R} - \gamma + \cos^2\alpha\right)\left(\ln\frac{2}{K} + \gamma + 1\right)\right],$$

$$\frac{1}{2\pi}\int_0^1 \left\{\left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma\right)\hat{f}_1(\xi) + \left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma + 1\right)\hat{f}_2(\xi)\cot\alpha\right\} d\xi$$

$$= -\frac{4k\cos^2\alpha}{\pi^2}\cdot\frac{T-1}{T^2 - \cos^2\alpha}\left(\ln\frac{2}{k} + \gamma + 1\right) \qquad (0 < x < 1). \tag{16}$$

The integration values of $\hat{f}_1$ and $\hat{f}_2$ over $0 < x < 1$, which are required for evaluation of the force, can be calculated directly from (16).

4. Slip effects on the drag and lift forces

The drag and lift forces are related to the distribution functions of Oseenlets $f_1(x)$ and $f_2(x)$ respectively [9]. Taking the conventional definitions of the drag and lift coefficients, we have

$$C_D = \frac{4}{R}\int_0^1 f_1(x)\,dx, \quad C_L = \frac{4}{R}\int_0^1 f_2(x)\,dx. \tag{17}$$

Integrating (14) correctly to $O(k)$ yields

$$\int_0^1 f_{1u}(x)dx = 2\pi \frac{T-\cos^2\alpha}{T^2-\cos^2\alpha}\left\{1-\frac{4k\cos^2\alpha}{\pi^2}\cdot\frac{T-1}{T-\cos^2\alpha}\left(\ln\frac{2}{k}+\gamma+1\right)\right\},$$

$$\int_0^1 f_{2u}(x)dx = \frac{2\pi\sin\alpha\cos\alpha}{T^2-\cos^2\alpha}\left\{1+\frac{4k}{\pi^2}(T-1)\left(\ln\frac{2}{k}+\gamma+1\right)\right\}. \tag{18}$$

The first terms on the right-hand side represent the result for the no-slip case, and the second terms imply the corrections arising from the slip in the narrow edge regions. Integrating (16) with respect to x after multiplying it by $\{x(1-x)\}^{-1/2}$, we get

$$\int_0^1 \hat{f}_1(x)dx = \frac{8k\cos^2\alpha}{\pi}\cdot\frac{T-1}{T^2-\cos^2\alpha}\left(1-\pi\frac{T-1}{T^2-\cos^2\alpha}\right)\left(\ln\frac{2}{k}+\gamma+1\right),$$

$$\int_0^1 \hat{f}_2(x)dx = -\frac{8k\sin\alpha\cos\alpha}{\pi}\cdot\frac{T-1}{T^2-\cos^2\alpha}\left(1-\frac{\pi T}{T^2-\cos^2\alpha}\right)\left(\ln\frac{2}{k}+\gamma+1\right). \tag{19}$$

The corrections produced by small slip over the central part of the plate are thus comparable to those in the edge regions.

Substituting (15) into (17) together with (18) and (19), we finally get

$$C_D = \frac{8\pi(T-\cos^2\alpha)}{R(T^2-\cos^2\alpha)}\left\{1-\frac{4(T-1)^2\cos^2\alpha}{\pi(T-\cos^2\alpha)(T^2-\cos^2\alpha)}k\left(\ln\frac{2}{k}+\gamma+1\right)\right\},$$

$$C_L = \frac{4\pi\sin 2\alpha}{R(T^2-\cos^2\alpha)}\left\{1+\frac{4T(T-1)}{\pi(T^2-\cos^2\alpha)}k\left(\ln\frac{2}{k}+\gamma+1\right)\right\}. \tag{20}$$

The presence of $\ln k$ terms reflects a singular property of the flow at the leading and trailing edges. These formulae show that the slip decreases the drag but increases the lift by amount of $O(k|\ln k|)$.

The Knudsen number is related to the Reynolds number and the speed ratio S, that is, the ratio of the uniform velocity to the most probable molecular speed [12]:

$$k = \lambda S/R, \tag{21}$$

where $\lambda = \frac{16}{5}\pi^{-1/2}$ for spherical molecules. Using this relation, we can express the drag and lift

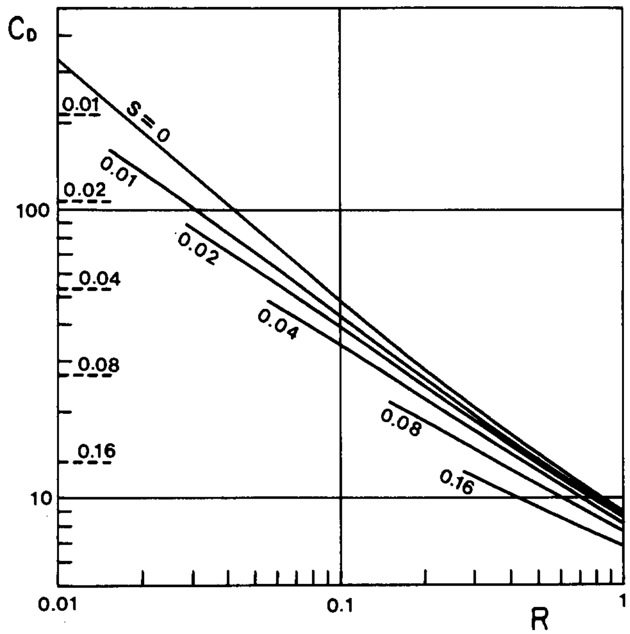

Figure 2. Variation of drag coefficient with Reynolds number ($\alpha = 30°$). —— present result; – – – – – free molecule flow.

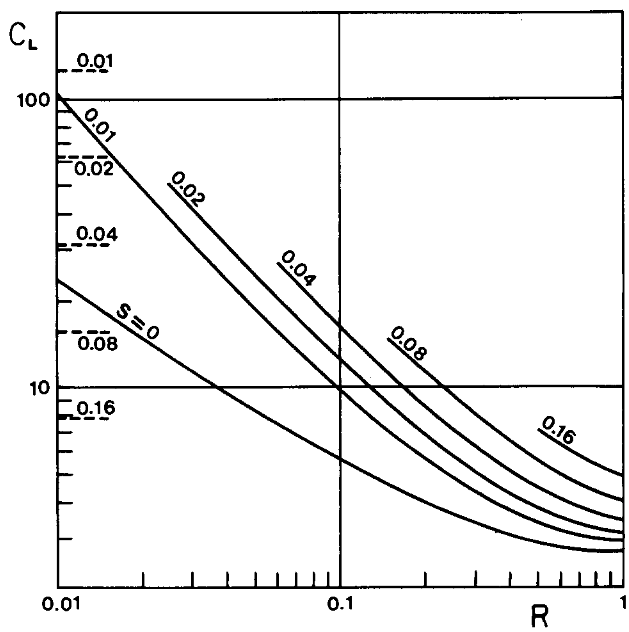

Figure 3. Variation of lift coefficient with Reynolds number ($\alpha = 30°$). —— present result; – – – – – free molecule flow.

coefficients as functions of R and S. Figures 2 and 3 show respectively the variations of C_D and C_L with R for values of $S \leqq 0.16$ and $\alpha = 30°$. The limit values for the free molecule flow ($k \to \infty$) obtained from the kinetic theory [2] are also indicated by the dashed lines for comparison. The curves of C_D and C_L deviate from the results for the no-slip flow ($S = 0$) as the Reynolds number decreases, showing part of transition to the free molecule flow. The drag coefficient becomes small as S increases at a fixed R of any value. On the other hand, the lift coefficient increases with S at a fixed R in a range of slip flow but takes a smaller value for larger S in the limit of free molecule flow. Therefore, we may conclude that the lift coefficient increases first, attains a maximum and then drops to a limit value as S increases at a fixed R.

Variations of C_D and C_L with α for $R = 0.2$ and $S = 0.01$ ($k = 0.09$) are shown in Figure 4. The results for no-slip flow are also given in the same figure for comparison. Slip produces the largest effect on C_D at $\alpha = 0°$, where drag is smaller than the no-slip value by about 10%. Deviation of C_D from the no-slip curve reduces as an angle of attack becomes large. No difference due to slip can be seen at $\alpha = 90°$, for the shearing stress on the surface of the plate is zero in the no-slip solution for this angle. The lift coefficient takes a higher value than in the no-slip case by about 40% for any α in this example.

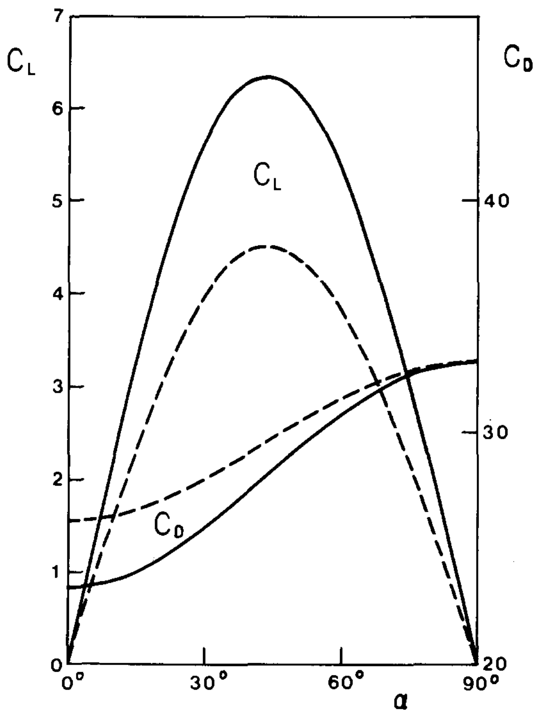

Figure 4. Variations of drag and lift coefficients with an angle of attack ($R = 0.2$). ——— $S = 0.01$; – – – – – no-slip flow ($S = 0$).

Appendix A

We put the integral equations (6) in such a form that Fourier transformation may be applied:

$$\int_{-\infty}^{\infty} \epsilon K_1(\epsilon\,|t-\tau|)\,\mathrm{sgn}\,(t-\tau)\,\{g_{1-}(\tau)-g_{2-}(\tau)\tan\alpha\}d\tau - 2\pi\frac{d}{dt}\{g_{1-}(t)-g_{2-}(t)\tan\alpha\}$$

$$= h_{1+}(t),$$

$$\int_{-\infty}^{\infty} \epsilon K_1(\epsilon\,|t-\tau|)\mathrm{sgn}\,(t-\tau)\,\{g_{1-}(\tau)+g_{2-}(\tau)\cot\alpha\}d\tau = h_{2+}(t) \qquad (\epsilon\to+0), \qquad \text{(A1)}$$

where

$$g_{j-}(t) = \begin{cases} g_j(t) & (t>0) \\ 0 & (t<0) \end{cases}, \qquad h_{j+}(t) = \begin{cases} 0 & (t>0) \\ h_j(t) & (t<0) \end{cases} \qquad (j = 1,2). \qquad \text{(A2)}$$

Both h_1 and h_2 are unknown functions at this stage. A convergence factor ϵ has been introduced for the existence of the Fourier transform and the limit of $\epsilon\to 0$ should be taken after the solution is obtained. The Fourier transform of (A1) is

$$2\pi ip\,\Lambda_1(p,\epsilon)\,\{G_{1-}(p)-G_{2-}(p)\tan\alpha\} = -H_{1+}(p),$$

$$\pi ip\,\Lambda_2(p,\epsilon)\,\{G_{1-}(p)+G_{2-}(p)\cot\alpha\} = -H_{2+}(p), \qquad \text{(A3)}$$

where

$$G_{j-}(p) = \int_{-\infty}^{\infty} g_{j-}(t)e^{-ipt}dt,\ H_{j+}(p) = \int_{-\infty}^{\infty} h_{j+}(t)e^{-ipt}dt \qquad (j = 1,2),$$

$$\Lambda_1(p,\epsilon) = 1+\tfrac{1}{2}(p^2+\epsilon^2)^{-1/2},$$

$$\Lambda_2(p,\epsilon) = (p^2+\epsilon^2)^{-1/2}. \qquad \text{(A4)}$$

Here the subscripts $\pm$ indicate the regularity of the functions in the upper and lower halves of the p-plane respectively. The Wiener-Hopf technique requires that $\Lambda_1(p,\epsilon)$ and $\Lambda_2(p,\epsilon)$ be factored into the form

$$\Lambda_j(p,\epsilon) = \Lambda_{j-}(p,\epsilon)/\Lambda_{j+}(p,\epsilon) \qquad (j = 1,2), \qquad \text{(A5)}$$

so that (A3) reduces to

$$ip\Lambda_{1-}(p,\epsilon)\,\{G_{1-}(p)-G_{2-}(p)\tan\alpha\} = -\Lambda_{1+}(p,\epsilon)H_{1+}(p)/2\pi,$$

$$ip\Lambda_{2-}(p,\epsilon)\,\{G_{1-}(p)+G_{2-}(p)\cot\alpha\} = -\Lambda_{2+}(p,\epsilon)H_{2+}(p)/\pi. \qquad \text{(A6)}$$

Factorization for $\Lambda_1(p, \epsilon)$ was performed by means of a standard procedure to yield [6]

$$\Lambda_{1-}(p,0) = \left(1 + \frac{1}{2|p|}\right)^{1/2} \exp\left[\frac{i}{2\pi}\operatorname{sgn}(p)\{P(|p|) - \tfrac{1}{2}\pi^2\}\right] \tag{A7}$$

for real p, where

$$P(p) = \int_0^p \frac{\ln(2s)}{s^2 - \frac{1}{4}}\, ds. \tag{A8}$$

The function $\Lambda_2(p, \epsilon)$ is easily factored and we get

$$\Lambda_{2-}(p,0) = p^{-1/2}. \tag{A9}$$

The left-hand side of the equations of (A6) are the analytic continuations of the functions on the right-hand side. Therefore, they represent two entire functions which are shown to be constants A and $Be^{(\pi/4)i}$ (say) to ensure the existence of Fourier inversions. As a result we have

$$G_{1-}(p) = -i\left\{\frac{A\cos^2\alpha}{p\Lambda_{1-}(p,0)} + \frac{B\sin^2\alpha}{p^{1/2}}e^{(\pi/4)i}\right\},$$

$$G_{2-}(p) = i\sin\alpha\cos\alpha\left\{\frac{A}{p\Lambda_{1-}(p,0)} - \frac{B}{p^{1/2}}e^{(\pi/4)i}\right\}. \tag{A10}$$

Fourier inversions of (A10) together with (5) and (7) yield the leading-edge solution (8).

Appendix B

Substituting (15) together with (14) into (4), we can rewrite the integral equations in the form

$$\frac{1}{2\pi}\int_0^1\left\{\left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma - 1\right)\hat{f}_1(\xi) - \left(\ln|x-\xi| - \ln\frac{4}{R} + \gamma\right)\hat{f}_2(\xi)\tan\alpha\right\}d\xi$$

$$= k[\{f_{1u}(x) + \hat{f}_1(x)\} - \{f_{2u}(x) + \hat{f}_2(x)\}\tan\alpha] - \frac{1}{2\pi}\int_0^1\left[\left(\ln|x-\xi| - \ln\frac{4}{R}\right.\right.$$

$$\left.+\gamma - 1\right)\left\{f_{1e}(\xi) + f_{1e}(1-\xi) - \frac{2(T-\cos^2\alpha)}{T^2 - \cos^2\alpha}(\xi^{-1/2} + (1-\xi)^{-1/2})\right\} - \left(\ln|x-\xi|\right.$$

$$\left.\left. - \ln\frac{4}{R} + \gamma\right)\left\{f_{2e}(\xi) + f_{2e}(1-\xi) - \frac{2\sin\alpha\cos\alpha}{T^2 - \cos^2\alpha}(\xi^{-1/2} + (1-\xi)^{-1/2})\right\}\tan\alpha\right]d\xi,$$

$$\frac{1}{2\pi}\int_0^1 \left\{\left(\ln|x-\xi|-\ln\frac{4}{R}+\gamma\right)\hat{f}_1(\xi)+\left(\ln|x-\xi|-\ln\frac{4}{R}+\gamma+1\right)\hat{f}_2(\xi)\cot\alpha\right\}d\xi$$

$$= -\frac{1}{2\pi}\int_0^1\left[\left(\ln|x-\xi|-\ln\frac{4}{R}+\gamma\right)\left\{f_{1e}(\xi)+f_{2e}(1-\xi)-\frac{2(T-\cos^2\alpha)}{T^2-\cos^2\alpha}\right.\right.$$

$$\left.\times\left(\xi^{-1/2}+(1-\xi)^{-1/2}\right)\right\}+\left(\ln|x-\xi|-\ln\frac{4}{R}+\gamma+1\right)\left\{f_{2e}(\xi)+f_{2e}(1-\xi)\right.$$

$$\left.\left.-\frac{2\sin\alpha\cos\alpha}{T^2-\cos^2\alpha}\left(\xi^{-1/2}+(1-\xi)^{-1/2}\right)\right\}\cot\alpha\right]d\xi. \tag{B1}$$

The following equations hold approximately for small k [6]:

$$(2\pi k)^{-1/2}\int_0^1 d\xi\int_0^\infty \{p(p+\tfrac{1}{2})\}^{-1/2}\sin\left\{\frac{\xi}{k}p-\frac{1}{2\pi}P(p)+\frac{\pi}{4}\right\}dp$$

$$\doteqdot 2\left\{1-\frac{k}{\pi}\left(\ln\frac{2}{k}+\gamma+1\right)\right\} \tag{B2}$$

and

$$f_{ju}(x)\doteqdot f_{jm}(x) \qquad (j=1,2), \tag{B3}$$

$$(2\pi k)^{-1/2}\int_0^1 \ln|x-\xi|\,d\xi\int_0^\infty \{p(p+\tfrac{1}{2})\}^{-1/2}\sin\left\{\frac{\xi}{k}p-\frac{1}{2\pi}P(p)+\frac{\pi}{4}\right\}dp$$

$$\doteqdot 2\left\{\ln(1-x)+x^{1/2}\ln\frac{1+x^{1/2}}{1-x^{1/2}}-2\right\}-\frac{2k}{\pi}\left[-\frac{\pi^2}{x^{1/2}}+\left(\ln\frac{2}{k}+\gamma+1\right)\right.$$

$$\left.\times\left\{\ln(1-x)+\frac{1}{x^{1/2}}\ln\frac{1+x^{1/2}}{1-x^{1/2}}\right\}-4\int_0^1\frac{\ln\eta}{1-x\eta^2}d\eta\right] \tag{B4}$$

in a central region of the plate where $x(1-x)\gg k$. Making use of these equations and the formula

$$\int_0^1 \xi^{-1/2}\ln|x-\xi|\,d\xi = 2\left\{\ln(1-x)+x^{1/2}\ln\frac{1+x^{1/2}}{1-x^{1/2}}-2\right\} \tag{B5}$$

in (B1), we get the equations (16) for small perturbations in the central region.

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